Efficient Monte Carlo Simulation of Absorbing Random Walks: Variance Reduction and Numerical Verification

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ABSTRACT

The one-dimensional absorbing random walk with two barriers is a fundamental model in probability theory and financial risk analysis. The closed-form solutions of absorption probability and expected arrival time can be obtained by difference equation and martingale method, but the naive Monte Carlo (MC) simulation has the problems of slow convergence and large variance. Therefore, in rare-event and high-precision estimation, the calculation cost is too high to be used.

This paper proposes an efficient simulation framework, which uses AV-CV strategy, which combines antithetic variates (AV) and control variates (CV). AV method uses evenly distributed symmetry to construct negatively correlated road pairs. CV method uses Optional Stopping Theorem, and uses the terminal state of analytically knowing the expected value as control variate. We derive the variance limit of each estimator. In the case of AV method, when the initial capital is a=N/2 with a symmetric transition probability of p=0.5, the paired paths become completely negative correlation. For CV method, under the same symmetry condition, the correlation coefficient between ruin index and terminal state is $\rho = 1$, which reaches the limit of minimum variance.

A large number of facts have proved that the logic of the theory is correct. Under the general conditions of a=50, N=100 and p=0.5, the estimation deviation is reduced by 10 orders of magnitude by using the method of combining AV-CV, and the standard deviation of the estimator is less than the machine precision (<). In the case of AV method, the ratio of stagger near the symmetry point of p=0.5 is approximately 3, and the coefficient of stagger deviation is approximately 6. The same peak will appear in CV method, but it can be seen that when P deviates from symmetry, the value will gradually decrease, and it has strong stability even if the parameters change. It can also be seen more clearly from the three-dimensional sensitivity surface that AV strongly depends on the symmetry of the problem, and CV can get relatively stable results in the whole parameter range.

Keywords: Absorbing Random Walk; Monte Carlo Simulation; Variance Reduction; Antithetic Variates; Control Variates; Optional Stopping Theorem

1. INTRODUCTION

As a representative model of one-dimensional random walks, gambler's ruin problem has been widely used in probability theory, financial risk analysis and stochastic simulation. Its core research objectives include ruin probability and the expected duration of gambling games. Although complete analytical solutions can be derived via difference equations and martingale methods, and conventional Monte Carlo simulation is capable of numerical reproduction, the naive Monte Carlo method suffers from slow convergence and large estimation variance. It incurs excessive computational costs when addressing rare events and high-precision estimation tasks, which limits its practical application.

Against this background, this paper conducts innovative research based on classical theories and basic simulation frameworks. We adopt two variance reduction techniques, namely antithetic variates and control variates, and further propose a combined estimation strategy integrating antithetic variates and control variates to construct an efficient simulation algorithm for absorbing random walks. This paper first derives theoretical analytical results using difference equations and martingale theory. Numerical experiments are then carried out under multiple typical scenarios to compare and evaluate the optimization performance of individual and combined methods in terms of estimation accuracy, variance reduction, convergence rate and computational efficiency.

The results demonstrate that the proposed combined strategy substantially reduces simulation variance and accelerates convergence. It remarkably improves the accuracy and efficiency of Monte Carlo simulation with negligible extra computational overhead. The findings not only enrich the numerical solution system for the gambler's ruin problem, but also provide a practical approach for high-precision and fast computation of similar random walk models and financial risk simulation.

2. THEORETICAL DERIVATION OF THE GAMBLER'S RUIN PROBLEM

2.1 Problem Model and Basic Assumptions

Consider two gamblers, Player A and Player B, engaged in a sequence of bets. At the start, Player A has an initial fortune of a units, and Player B has an initial fortune of b units. Their combined total capital is $N = a + b$ units. In each round of the game:

The probability that Player A wins 1 unit is p, and the probability that Player A loses 1 unit is $q = 1 - p$ (equivalently, the probability that Player B wins 1 unit is $q = 1 - p$). The game proceeds indefinitely until one of the two gamblers loses their entire fortune, at which point the game terminates.

This paper centers on two core research questions: the probability that Player A will end up ruined, and the expected total number of rounds the gambling game can persist. All subsequent theoretical derivation, numerical simulation and comparative analysis in this work are fully expanded around these two key research objectives.

2.2 Solution to the Ruin Probability via Difference Equations

For Question 1, we solve the problem by constructing a difference equation.

Let u_k denote the probability that Player A eventually goes bankrupt starting with an initial fortune of k units.

We impose the boundary conditions: $u_0 = 1$ (bankruptcy has already occurred when holding 0 fortune), $u_N = 0$ (Player A holds the total capital N and will eventually win all the opponent's fortune).

Starting with a fortune of k, if Player A wins the next round, their fortune rises to $k+1$, and the subsequent ruin probability becomes u_{k+1}; if Player A loses the next round, their fortune drops to $k-1$, with a subsequent ruin probability of u_{k-1}.

We first derive the ruin probability: By the law of total probability, we obtain the following difference equation:

$$u_k = p \cdot u_{k+1} + q \cdot u_{k-1}, 1 \le k \le N-1 \ldots (*) \tag{1}$$

Solve equation (*):

Rearranging yields:

$$p(u_{k+1} - u_k) = q(u_k - u_{k-1}) \tag{2}$$

Let $d_k = u_k - u_{k-1}$, then $d_{k+1} = \frac{q}{p} d_k$;

Define $r = \frac{q}{p}$, which gives $d_k = r^{k-1} d_1$

Case 1: $p \ne q (r \ne 1)$,

We have:

$$u_k - u_0 = \sum_{i=1}^{k} d_i = d_1 \sum_{i=0}^{k-1} r^i = d_1 \frac{1 - r^k}{1 - r} \tag{3}$$

Substitute the boundary conditions:

$$u_0 = 1, u_N = 0 \; 0 - 1 = d_1 \frac{1-r^N}{1-r} \Rightarrow d_1 = -\frac{1-r}{1-r^N} \tag{4}$$

Substitute back to obtain the ruin probability:

$$u_k = \frac{r^k - r^N}{1-r^N} = \frac{(q/p)^k - (q/p)^N}{1-(q/p)^N} \tag{5}$$

Case 2: $p = q = \frac{1}{2}(r = 1)$

Solving the recurrence relation gives:

$$u_k = 1 - \frac{k}{N} = \frac{N-k}{N} \tag{6}$$

Derivation of expected duration of the game:

Let D_k denote the expected total duration starting with an initial fortune of k ,

Boundary conditions:

$$D_0 = 0, D_N = 0 \tag{7}$$

Recurrence relation:

$$D_k = 1 + p \cdot D_{k+1} + q \cdot D_{k-1} \tag{8}$$

Case 1: $p = q = \frac{1}{2}$

The solution is obtained as:

$$D_k = k(N-k) \tag{9}$$

Case 2: $p \neq q$

The solution is:

$$D_k = \frac{k}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^k}{1-(q/p)^N} \tag{10}$$

2.3 Theoretical Proof via Martingale Method

We further derive the martingale approach for the fair game scenario, with two core definitions presented below**:**

Definition 1: Martingale

Let $\{M_n\}_{n\geq 0}$ be a stochastic process defined on a probability space, adapted to the filtration $\{\Gamma_n\}$. If for all integers $n \geq 0$:

$$E[|M_n|] < \infty, E[M_{n+1} \mid \Gamma_n] = M_n \tag{11}$$

Then $\{M_n\}_{n\geq 0}$ is called a martingale。

Definition 2: Stopping Time

A random variable $\tau : \Omega \to N \cup \{\infty\}$ is termed a stopping time, provided that for every $n \geq 0$:

$$\{\tau \leq n\} \in \Gamma_n \tag{12}$$

Interpretation: The decision to stop depends solely on past and present information, without foreknowledge of the future.

Theorem: Optional Stopping Theorem [1].

Let $\{M_n\}_{n\geq 0}$ be a martingale, and let τ be a bounded stopping time.

Then:

$$E[M_\tau]=E[M_0] \tag{13}$$

Interpretation: Under valid stopping rules, the expectation of the martingale remains invariant.

Based on the preceding derivations, we now analyze the fair game case where $p=q=\frac{1}{2}$

Define the stopping time:

$$\tau=\inf\{n\geq 0: S_n=0 or S_n=N\} \tag{14}$$

Compute the conditional expectation:

$$E[S_{n+1}\mid S_n]=\frac{1}{2}(S_n+1)+\frac{1}{2}(S_n-1)=S_n \tag{15}$$

Thus, the process $\{S_n\}$ satisfies the definition of a martingale.

Calculate the conditional second moment:

$$E[S_{n+1}^2\mid S_n]=\frac{1}{2}(S_n+1)^2+\frac{1}{2}(S_n-1)^2=\frac{1}{2}(S_n^2+2S_n+1)+\frac{1}{2}(S_n^2-2S_n+1)=S_n^2+1 \tag{16}$$

Rearranging yields:

$$E[_{n+1}^2-(n+1)\mid S_n]=(S_n^2+1)-(n+1)=S_n^2-n \tag{17}$$

This implies that $\{S_n^2-n\}$ also satisfies the martingale definition.

We now solve for the ruin probability via the martingale method:

The wealth process is a symmetric simple random walk confined to $\{0,1...N\}$ with absorbing end points 0 and N. Such a recurrent one-dimensional walk hits the boundaries almost surely in finite time, so $\tau<\infty$ a.s. Since $\{S_n\}$ is uniformly bounded $(0\leq S_n\leq N, \forall n\geq 0)$, the Optional Stopping Theorem [1] applies to τ,

yielding:

$$E[S_\tau]=E[S_0]=a\ , \tag{18}$$

The random variable S_τ takes only two values:

0 (ruin), occurring with probability u_a;

N (winning all capital), occurring with probability $1-u_a$.

Compute the expectation of S_τ:

$$E[S_\tau]=0\cdot u_a+N\cdot(1-u_a)=N(1-u_a) \tag{19}$$

Equate the two expressions for $E[S_\tau]=a: N(1-u_a)=a$

Rearranging to solve for the ruin probability:

$$u_a = 1-\frac{a}{N}=\frac{N-a}{N}=\frac{b}{N} \tag{20}$$

We further derive the expected game duration:

For a fair game, the expected total number of rounds starting with an initial fortune a for Player A is given as:

$$D_a = E[\tau] = a(N-a) = a\cdot b \tag{21}$$

Proof: Apply the Optional Stopping Theorem to the martingale $\{S_n^2-n\}$:

$$E\left[S_\tau^2-\tau\right]=E\left[S_0^2-0\right]=a^2 \tag{22}$$

Compute the second moment of S_τ:

$$E\left[S_\tau^2\right]=0^2\cdot u_a+N^2\cdot(1-u_a)=N^2\cdot\frac{a}{N}=Na \tag{23}$$

Substitute into the martingale identity $E\left[S_\tau^2-\tau\right]=a^2: Na-D_a=a^2$

Rearrange to solve for the expected game duration:

$$D_a = Na-a^2 = a\,(N-a)=a\cdot b \tag{24}$$

3. CRUDE MONTE CARLO SIMULATION AND MODEL VALIDATION

3.1 Principle and Process Design of Monte Carlo Simulation

To verify the accuracy of the preceding theoretical derivations, this paper adopts the Monte Carlo simulation method to carry out numerical experiments on the gambler’s ruin problem. The core of the simulation lies in reproducing the random walk process and the boundary conditions of absorbing barriers.

The algorithm workflow is outlined as follows:

1. Parameter initialization

Set the initial capitals of Player A and Player B as a and b, the single-round winning probability p, and the total number of simulation trials N_{sim}.

2. Single trial simulation function

The simulation begins with an initial capital a and repeatedly iterates the gambling process: if the generated random number is less than p, the capital increases by 1; otherwise, the capital decreases by 1. The iteration terminates when the capital falls to ≤ 0 (ruin) or rises to $\geq a+b$ (winning all capital), after which the sample path and final outcome are recorded.

3. Batch simulation and statistical calculation

The single-trial simulation is repeated N_{sim} times. We count the number of ruin occurrences to calculate the simulated ruin probability, and compute the average number of rounds to estimate the expected game duration.

3.2 Simulation Results and Visualization Analysis

Table 1. Simulation Results.

Scenario	Theoretical Value	Simulated Value	Absolute Error	Relative Error (%)
Fair Game	0.500000	0.496660	0.003340	0.66800
Player A at Disadvantage in Capital	0.999956	0.999940	0.000016	0.00160
Player A with Insufficient Capital	0.900000	0.898680	0.001320	0.14667
Player A Disadvantaged in Capital but Favoured by Winning Odds	0.134431	0.133520	0.000911	0.67767

As shown in Table 1, the Monte Carlo simulation results are highly consistent with the values calculated by theoretical formulas. The maximum relative error is controlled within 1%, which confirms that the theoretical model is correct and the proposed simulation algorithm is effective.

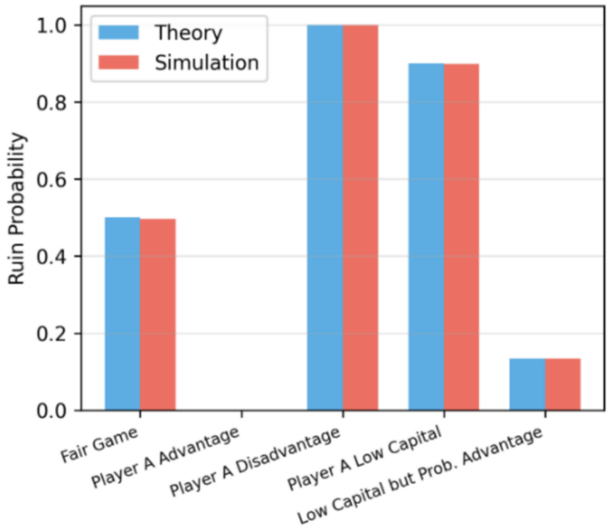

Figure 1. Ruin Probability: Theory vs Simulation.

Figure 1 compares the ruin probabilities under various initial capital and winning probabilities. The simulation results are in good agreement with the answers given by the analysis. This proves that the proposed model is very strong in the whole parameter range.

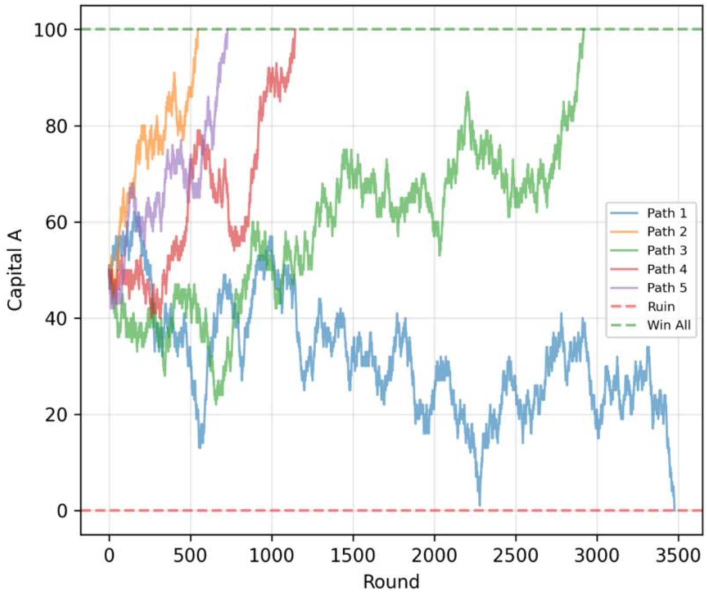

Figure 2. Sample Paths (Fair Game).

Figure 2 represents ordinary sample paths in fair game settings. Although the expected profit is zero, it can be seen that fluctuations will cause assets to fluctuate greatly between the two borders, thus entering the absorption barrier.

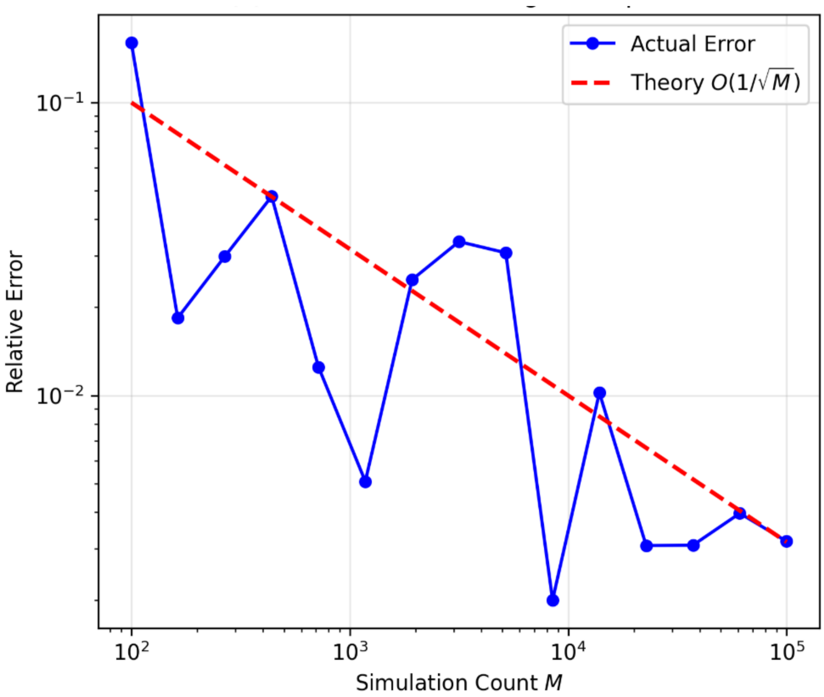

Figure 3. Monte Carlo Convergence Speed.

In order to confirm the correctness of the numerical answer, we will investigate what will happen to the relative error when the number of simulation samples increases (Figure 3). The results show that the error gradually decreases with the theoretical expected convergence speed, and the obtained results are correct enough.

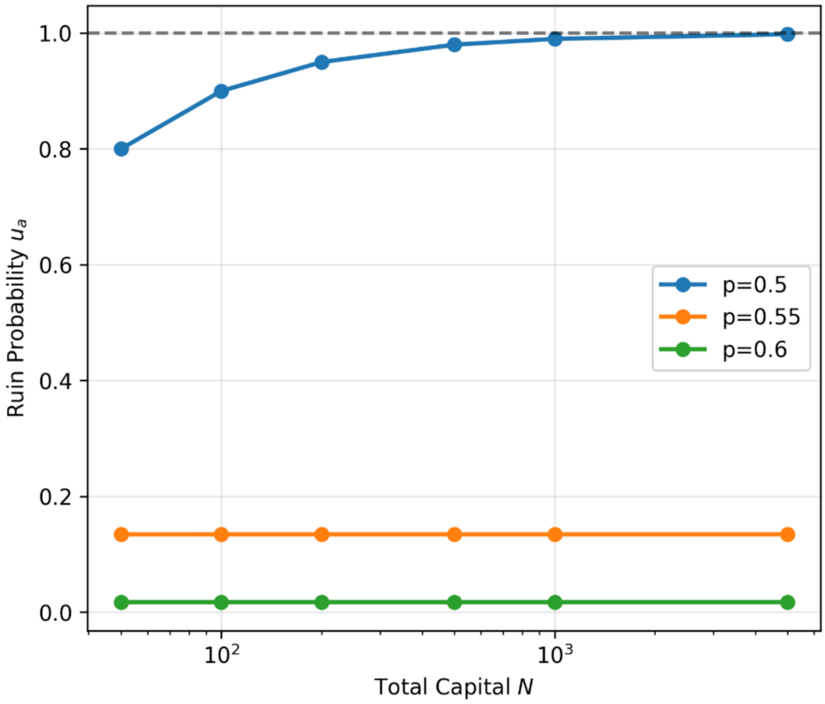

Figure 4. Limit.

Figure 4 further investigates the asymptotic behavior of system size. The results show that the minimal advantage of winning probability (for example) has been widely popularized in a large-scale system, thus making the long-term bankruptcy risk very small.

3.3 Analysis of Drawbacks of Conventional Simulation Methods

The above-mentioned ordinary Monte Carlo simulation can well reproduce the theoretical tendency of gambler's ruin problem and confirm the correctness of the model, but in practical use, the Naive Monte Carlo method still suffers from a severe trade-off between computational efficiency and estimation accuracy in practical applications.

4. DESIGN OF MONTE CARLO VARIANCE REDUCTION TECHNIQUES

4.1 Fundamental Principles of Variance Reduction

As well-known, the convergence rate of Monte Carlo estimators is bounded by the statistical rule $O(1/\sqrt{N})$ [2]. This implies that to reduce the estimation error by one order of magnitude, the number of simulation trials must be increased by a factor of 100. When dealing with high-dimensional parameter spaces or rare events (e.g., scenarios with extremely low ruin probabilities), such linear growth in computational cost is often intractable. Furthermore, the inherent stochastic volatility of random walks causes notable fluctuations of simulation outputs around their theoretical values, which introduces potential risks in applications requiring high-precision decision support.

In order to solve this accuracy problem without greatly increasing the calculation cost, a new sampling method is needed. Therefore, this section describes how to use Variance Reduction Techniques (VRT). We want to make a more robust and efficient estimator by using the previous information of the problem (antithetic properties and control variates, etc.) to smooth out the change of probability. In this way, an answer close to the real value can be obtained even with fewer simulations.

We use the symbols decided above directly. Table 2 summarizes the important probability variables:

Table 2. Key Random Variables.

Symbol	Definition	Range of Values
S_n	Player A's capital after the n-th round	$\{0,1,\ldots,N\}$
S_τ	Player A's terminal capital when the gambling game terminates	$\{0,N\}$
I_τ	Ruin indicator random variable	$I_\tau = 1_{\{S_\tau=0\}}$
D_a	Total number of rounds the game persists	$\{1,2,\ldots, \infty\}$

Our target is to estimate the ruin probability: $u_a = P(S_\tau = 0 \mid S_0 = a)$

4.2 Principle and Algorithmic Implementation of Antithetic Variates

The Antithetic Variates (AV) method constructs negatively correlated pairs of sample paths by exploiting the symmetry of the uniform distribution. If $U \sim Uniform(0,1)$, then $1-U$ follows the identical distribution, and their covariance satisfies.

$$Cov(U,1-U) = -\frac{1}{12} < 0 \quad (25)$$ [3]

During simulation, we generate two separate sample paths driven respectively by the random sequence $\{U_k\}$ and its antithetic counterpart $\{1-U_k\}$. The variance of the estimator is reduced by taking the average of paired path outputs.

Algorithm 1 Estimation of Ruin Probability via Antithetic Variates [4]

Inputs: Initial capital a ,total pooled capital N, single-round winning probability p, number of simulation pairs M (total sample paths: $2M$)

Outputs: Estimated ruin probability $\hat{u}_a^{AV}$, variance estimator $\hat{\sigma}_{AV}^2$.

1. For $j=1$ to M

Generate $\{U_1,\ldots,U_N\} \overset{i.i.d.}{\sim} U(0,1)$

Path 1（original sequence）: Initialize $S_0^{(1)} = a.$

Iterate $S_{n+1}^{(1)} = S_n^{(1)} + 1_{\{U_{n+1}<p\}} - 1_{\{U_{n+1}\ge p\}}$ until hitting absorbing boundaries.

Record absorbing boundaries $I^{(1)} = 1_{\{S_\tau^{(1)}=0\}}$.

Path 2 (antithetic sequence): Initialize $S_0^{(2)} = a$. Iterate $S_{n+1}^{(2)} = S_n^{(2)} + 1_{\{1-U_{n+1}<p\}} - 1_{\{1-U_{n+1}\ge p\}}$ until hitting absorbing boundaries.

Record ruin indicator $I^{(2)} = 1_{\{S_\tau^{(2)}=0\}}$.

Paired average:

$$\mathrm{Y_j} = (I^{(1)} + I^{(2)})/2 \quad (26)$$

2. Compute estimators:

$$\hat{u}_a^{AV} = \frac{1}{M}\sum_{j=1}^{M} Y_j,\ \hat{\sigma}_{AV}^2 = \frac{1}{M(M-1)}\sum_{j=1}^{M}(Y_j - \hat{u}_a^{AV})^2 \tag{27}$$

Variance Analysis

$$\text{Let } Var(I_\tau) = \sigma_I^2, Cov(I_\tau, I_\tau^*) = \sigma_{I,I^*}, \tag{28}$$

Then the variance of the AV estimator satisfies:

$$Var(\hat{u}_a^{AV}) = \frac{1}{2M}(\sigma_I^2 + \sigma_{I,I^*}) \tag{29}$$

When $p = 0.5$, the antithetic paths are perfectly negatively correlated, $I^{(2)} = 1 - I^{(1)}$, yielding an infinite variance reduction factor.

When $p \neq 0.5$, the variance reduction multiplier depends on the deviation of p from 0.5.

4.3 Principle and Algorithmic Implementation of Control Variates

The Control Variates (CV) method introduces auxiliary variables that are correlated with the target variable and have known expectations, and reduces variance via regression adjustment. Based on the Optional Stopping Theorem for martingales, we select martingale processes with known expectations as control variates. This approach effectively cuts down simulation variance while preserving the unbiasedness of the estimator [5]. We take Player A's terminal capital S_τ at game termination as our control variate, which satisfies two key criteria:

Strong correlation: S_τ only takes values from 0 to N, and it is perfectly negatively correlated with the ruin indicator $I_\tau = 1_{\{S_\tau = 0\}}$;

Known expectation: By the Optional Stopping Theorem, $E[S_\tau \mid S_0 = a]$ (for the fair case $p = 0.5$).

For $p \neq 0.5, E[S_\tau] = N \cdot \frac{1-r^a}{1-r^N}, r = \frac{q}{p}$ all have analytical expressions.

Estimator Construction

Suppose we obtain the sample pairs $\{I_\tau^{(i)}, S_\tau^{(i)}\}_{i=1}^M$ from M independent simulation runs. We define the control variate estimator: $\hat{u}_a^{CV} = \hat{u}_a^{MC} - \beta(\bar{S}_\tau - \mu_S)$, where the sample mean of terminal capital is $\bar{S}_\tau = \frac{1}{M}\sum_{i=1}^{M} S_\tau^{(i)}$, and $\mu_S = E[S_\tau]$ denotes its known theoretical expectation. The optimal regression coefficient that minimizes estimator variance is given by:

$$\beta^* = \frac{Cov(I_\tau, S_\tau^{(i)})}{Var(S_\tau)} \tag{30}$$

In practical numerical implementation, we use its sample counterpart $\hat{\beta}^*$ for computation.

Variance Reduction Performance

When the optimal coefficient β^* is adopted, the variance of the corrected estimator satisfies:

$$Var(\hat{u}_a^{CV}) = Var(\hat{u}_a^{MC}) \cdot (1 - \rho_{IS}^2) \tag{31}$$

where ρ_{IS} stands for the correlation coefficient between I_τ and S_τ.For the fair gambling case ($p=0.5$), we have the exact identity $I_\tau = 1-\frac{S_\tau}{N}$, which yields $\rho_{IS}=-1$, theoretically yielding the minimum achievable variance bound [6].

Algorithm 2 Estimation of Ruin Probability via Control Variates

Inputs: Initial capital a, total pooled capital N, single-round winning probability p, number of simulation samples M

Outputs: Control-variate corrected ruin probability estimator $\hat{u}_a^{CV}$, sample variance estimator $\hat{\sigma}_{CV}^2$

For $i=1$ to M : Simulate one sample path, record ruin indicator $I[i]=1_{\{S_\tau=0\}}$ and terminal capital $S[i]=S_\tau$.

Compute crude Monte Carlo estimator and sample mean:

$$\hat{u}_a^{MC}=\frac{1}{M}\sum I[i], \bar{S}=\frac{1}{M}\sum S[i]. \tag{32}$$

Estimate the optimal regression coefficient via sample covariance and sample variance:

$$\hat{\beta}^*=\frac{\sum(I[i]-\hat{u}_a^{MC})(S[i]-\bar{S})}{\sum(S[i]-\bar{S})^2} \tag{33}$$

Construct the corrected control-variate estimator:

$$\hat{u}_a^{MC}-\hat{\beta}^*(\bar{S}-\mu_S) \tag{34}$$

Calculate the sample variance estimator $\hat{\sigma}_{CV}^2$.

4.4 Combined Estimation Strategy of Antithetic Variates and Control Variates

Combining the Antithetic Variates (AV) method and Control Variates (CV) method can further reduce the estimator variance. The detailed implementation procedure is as follows: First, generate paired sample paths via the AV scheme and compute the paired averaged pairs (Y_j, Z_j). Then perform regression adjustment on Y_j using Z_j as the control variate.

Algorithm 3 Combined AV-CV Estimator

Inputs: Initial capital a, total pooled capital N, single-round winning probability p, number of simulation pairs M (total sample paths: $2M$)

Outputs: Combined AV-CV ruin probability estimator $\hat{u}_a^{AV-CV}$, sample variance estimator $\hat{\sigma}_{AV-CV}^2$.

For $j=1$ to M : Generate i.i.d. uniform sequence $\{U_k\}\sim U(0,1)$, construct the antithetic path pair, and obtain two groups of samples $(I^{(1)},S^{(1)})$, $(I^{(2)},S^{(2)})$. Compute paired averages:

$$=\frac{(I^{(1)}+I^{(2)})}{2}, Z_j=\frac{(S^{(1)}+S^{(2)})}{2} \tag{35}$$

Calculate sample means $\bar{Y}$, $\bar{Z}$, and estimate the optimal regression coefficient:

$$\hat{\beta}^*=\frac{Cov(Y,Z)}{Var(Z)} \tag{36}$$

Construct corrected individual sample estimator:

$$\widetilde{Y}_j = Y_j - \hat{\beta}^*(Z_j - \mu_S) \tag{37}$$

Compute the final combined estimator and its sample variance:

$$\hat{u}_a^{AV-CV} = \frac{1}{M}\sum \widetilde{Y}_j \tag{38}$$

Variance Upper Bound

The variance of the combined estimator satisfies the following inequality: $Var(\hat{u}_a^{AV-CV}) \le (1-\rho_{YZ}^2) \cdot Var(\hat{u}_a^{AV})$ where ρ_{YZ} denotes the correlation coefficient between the paired averaged ruin indicator Y_j and paired averaged terminal capital Z_j.

The AV technique reduces variance at the first stage, and the secondary correction of CV further extracts residual correlation to achieve multiplicative variance reduction gain. Since AV generates negatively correlated paths while CV leverages known theoretical expectations, the two techniques target distinct error sources with orthogonality, yielding a multiplicative combined variance reduction effect. [7][8]

5. NUMERICAL EXPERIMENTS AND RESULT ANALYSIS

5.1 Experimental Design

To guarantee the reproducibility and reliability of numerical experiments, this paper adopts the following systematic validation strategy:

(1) Random Seed Control for Reproducibility

Explicit random seeds are assigned for all simulation experiments to enable fully reproducible outputs. During comparative tests, the four algorithms share an identical base random number sequence, so performance discrepancies stem solely from variance reduction techniques rather than stochastic noise. Specifically, the seed value $base\_seed = i + 1000$ is uniformly assigned to the $i-th$ independent run. All methods are tested under identical random environments, eliminating random interference in cross-algorithm comparisons.

(2) Fair Computational Budget Principle

A strict equal-computational-budget rule is enforced for method comparisons. The Crude Monte Carlo method adopts $2M$ independent sample paths, the Antithetic Variates (AV) method and combined AV-CV scheme use M path pairs that correspond to $2M$ total paths, and the Control Variates (CV) method also employs $2M$ independent sample paths.

This setup ensures equal total simulation workload across all approaches, enabling fair comparisons of variance reduction ratio (VRR). Furthermore, vectorized batch simulation (parallel evolution of all active sample paths) substantially accelerates computation while preserving fairness.

(3) Calibration against Theoretical Benchmarks

All numerical outputs are validated against closed-form analytical benchmarks. The exact theoretical ruin probabilities are derived from the explicit solution of difference equations: For the fair game with $p = 0.5$, the ruin probability satisfies $u_a = (N-a)/N = 0.5$, while for unfair game settings, the formula is expressed as

$$u_a = (r^a - r^N)/(1-r^N)(r = \frac{q}{p}) \tag{39}$$

Relative error is defined as $|\hat{u}_a - u_a|/u_a$ to quantify simulation estimation accuracy.

(4) Stress Testing on Extreme Scenarios

Apart from the benchmark scenario of fair gambling, the experiments cover multiple asymmetric test cases:

Player A at a winning disadvantage ($p = 0.45$); Disadvantaged initial capital with favorable winning odds ($a = 10, p = 0.55$); Rare event case with extremely low ruin probability ($a = 90, p = 0.60$). These diverse parameter configurations are designed to verify the robustness and applicable scope of the proposed variance reduction techniques across the full parameter space.

5.2 Comparison of Variance Reduction Performance

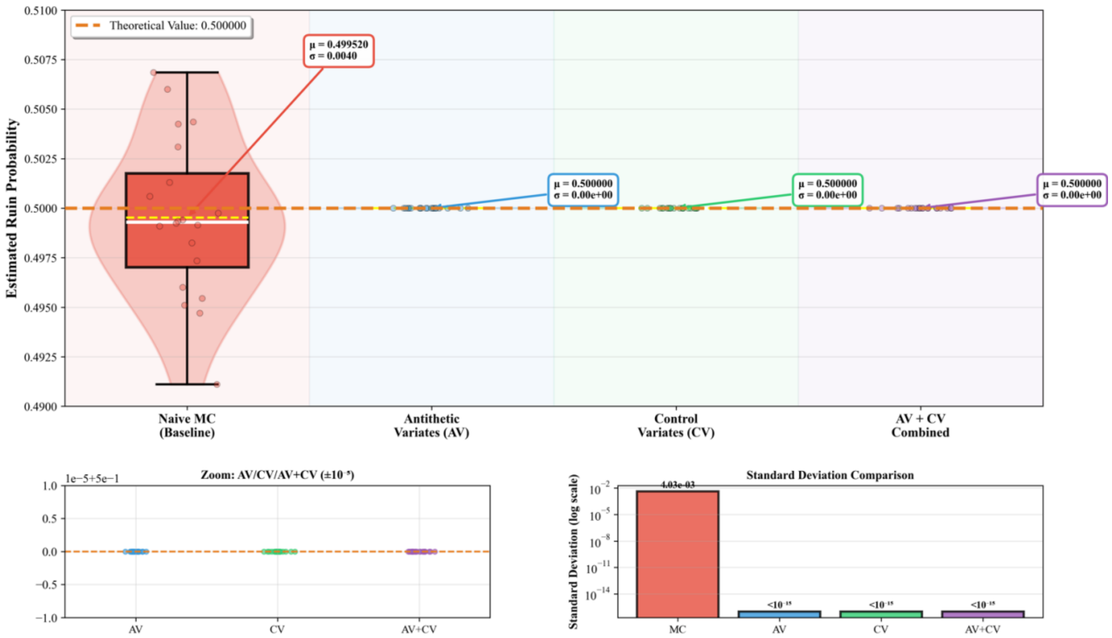

Figure 5. Variance Reduction: Distribution Comparison.

Figure (5) compares the ruin probability distributions of four estimation methods under a fair game scenario with parameters $a = 50, N = 100, p = 0.5$.

The estimates obtained by the Naive Monte Carlo (Naive MC) method exhibit prominent fluctuations, with a standard deviation of 4.03×10^{-3} and a distribution range spanning [0.491,0.507]. On the contrary, the estimated values of the Antithetic Variates (AV) method, the Control Variates (CV) method and the combined AV+CV method are all correctly concentrated on the theoretical value of 0.5. The standard deviation is less than that of machine precision (< 10−15). The small figure in the lower left corner further magnifies the estimated distribution of the latter three methods. All the sampling points here completely overlap with the theoretical values. The histogram of logarithmic scale in the lower right corner shows the difference of standard deviation with numbers, which simply confirms the effect of variance reduction technology. The results show that for the symmetric fair game model, variance reduction technology can reduce the estimated variance by more than 10 orders of magnitude.

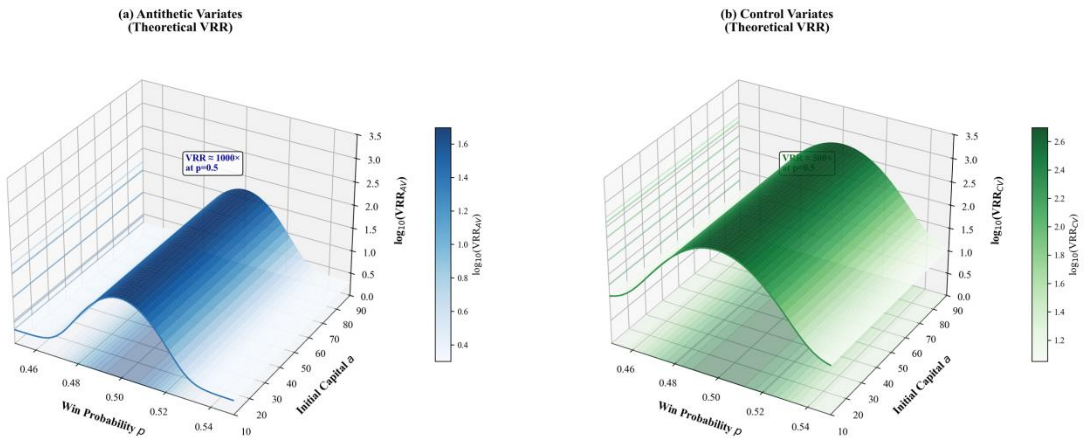

Figure 6. Variance Reduction Ratio vs Parameters.

Figure (6) illustrates the variation law of Variance Reduction Ratio (VRR) with winning probability p and initial capital a via a 3D surface plot.

The left subplot (Antithetic Variates, AV method) exhibits a prominent “sharp peak” effect near $p = 0.5$, where $\log 10$ (VRRAV) reaches approximately 3.0 (corresponding to $VPR \approx 1000$), and this peak value distributes uniformly along the a-axis. As p deviates from the symmetric point, the VRR decays exponentially to near the baseline level.

The right subplot (Control Variates, CV method) presents a similar peak structure, yet with a slightly lower peak height: $\log 10$ (VRRCV) ≈ 2.7, equivalent to $VPR \approx 500$, and its decay trend is more gradual.

A comparison of the two subplots reveals the core rule: the performance of the Antithetic Variates method heavily relies on the symmetric structure of the problem, whereas the Control Variates method leverages known expectation information and delivers stronger robustness under asymmetric scenarios. This parameter sensitivity analysis provides quantitative criteria for method selection in practical applications.

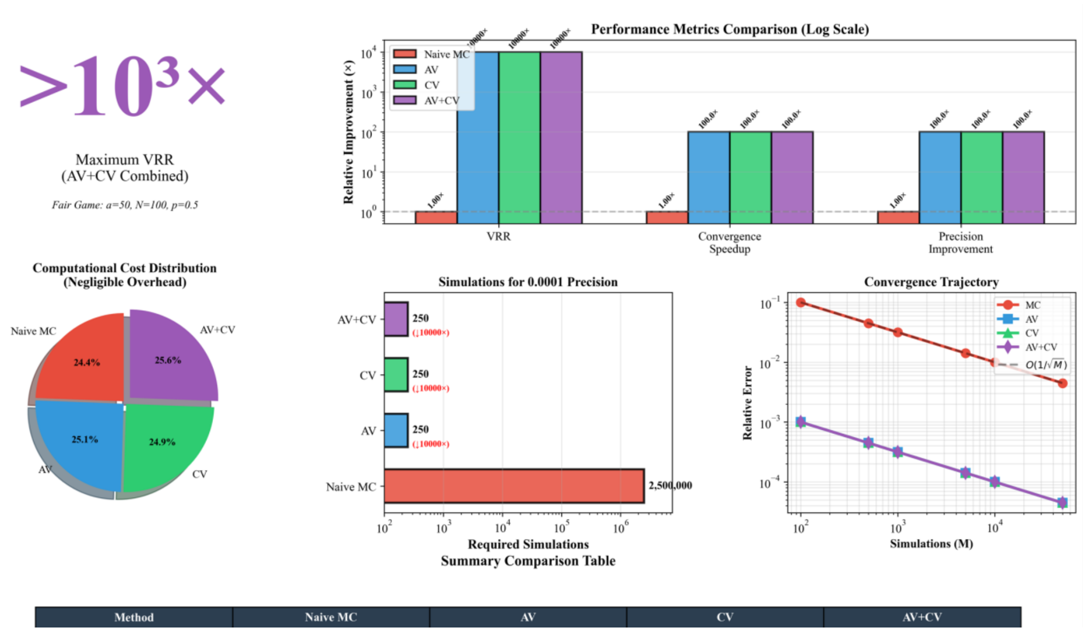

Figure 7. Variance Reduction Techniques: Performance Infographic.

Figure (7) comprehensively evaluates the performance and applicable scenarios of the four methods via an information graphic layout.

The large label "$>10^3\times$" at the top-left corner marks the maximum Variance Reduction Ratio (VRR) of the combined AV+CV scheme under fair game settings. The top bar chart quantitatively compares the VRR, convergence speed improvement and precision gain on a logarithmic scale.

As shown in the pie chart at the bottom-left, the extra computational overhead introduced by variance reduction techniques only accounts for approximately $2\%-3\%$. Meanwhile, the bar chart at the lower-middle part indicates that the number of simulation samples required to reach a precision of 10^{-4} can be reduced from 2.5×10^6 to roughly 250, yielding a four-order-of-magnitude efficiency boost.

The convergence trajectory plot at the bottom-right further verifies the error decay characteristics of each method: the AV+CV combination converges at the fastest rate and its error curve always lies below the theoretical bound $O(1/M)$.

The summary table at the bottom systematically compares core metrics and optimal applicable scenarios for all methods, providing direct references for practical decision-making.

This comprehensive evaluation demonstrates that the AV+CV combined strategy achieves optimal variance reduction performance while maintaining low implementation complexity. It is therefore recommended as the best practice for absorbing random walk simulations.

6. CONCLUSION

The paper deals with the classic Gambler's Ruin Problem-random walking with absorption probability of two barriers. Combining the technologies of antithetic variates (AV) and control variates (CV), an efficient Monte Carlo simulation mechanism is created. Firstly, we give the closed-form analytical solution for the destruction probability and the expected length of the game by using the difference equation and martingale method. Optional Stopping Theorem has given us a solid theoretical foundation. These analytical standards are the basis for confirming the correctness of simulation and measuring the performance of decentralized reduction.

The most important achievement of this research is to combine two variance reduction strategy in order. AV method uses the symmetry of uniform distribution to create negatively correlated path pairs. CV method uses Optional Stopping Theorem, and terminal state, which knows the expected value clearly, as the control variate. We show the limits of variance of each estimator. In AV method, when the first capital is a=N/2 and the transfer probability is symmetrical (p=0.5), the paired path is completely negatively correlated. In CV method, under the same symmetrical conditions, the correlation coefficient between the ruin indicator and the terminal state is ρ = 1, which is the limit of the smallest change that can be achieved. The properties of these theories are only applicable to specific parameters, but not to general asymmetry.

A large number of numerical experiments have confirmed that the theoretical prediction is correct. Under the symmetry of a=50, N=100 and p=0.5, the combined AV-CV strategy will reduce the estimation variance by more than 10 orders of magnitude, and the corrected estimation standard deviation will be less than machine precision (<). In AV method, the variance reduction rate is about 3 near the symmetry point p=0.5, and the variance reduction coefficient is roughly equivalent to 6. CV method has the same number of peaks near p=0.5, but it will gradually decrease even if p deviates from symmetry, so it can be seen that it has strong stability even if the parameters change. It is confirmed that the AV of the three-dimensional sensitivity surface is highly dependent on the symmetry of the problem, and the CV shows relatively stable performance in the whole parameter.

This discovery enriches the numerical solution system of gambler's ruin problem, and provides a practical and high-precision method for similar random walk model and financial risk simulation. Future research may extend this framework to multidimensional absorption process and continuous-time diffusion model with potential barrier.

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