

From Constraint Counting to Geometry-Informed Frameworks: A Critical Review of Mobility Theory for Overconstrained Mechanisms

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ABSTRACT

In mechanisms, there is a growing trend to establish theories, aiming to explain the mobility of mechanism. According to the degree-of-freedom (DOF) theory, multiple mechanism designs have been built, for example, Bennett and Bricard linkages. However, present theories of mobility are insufficient for all mechanisms. To broaden the application scope of the DOF theory, researchers have tried to introduce mathematic theories on DOF calculation to satisfy present mechanism design. This paper is a detailed summary of the historical changes and recent progress of DOF theory. In particular, it focuses on the analysis of overconstrained mechanisms, which is the standard to judge whether various theoretical methods are correct or not. By tracing the changes of existing methods, this review not only shows their development process, but also discusses their inherent limitations in dealing with complex machining.

Keywords: Mobility Theory, Overconstrained Mechanisms, Screw Theory

1. INTRODUCTION

The beginning of DOF theory can be traced back to the important research of Grübler in 1884. Grübler originally proposed this as a framework for understanding the mobility of mechanisms. In this study, the mobility of mechanism is defined as DOF, and a strong formula for calculating DOF of flat closed kinematic chains is given ^[1]. In the 20th century, a great deal of research was done, trying to create a widely used method to calculate the difficulty of any rigid body mechanism. Multiple approaches were exhibited in literature by Koeings ^[2], Malystsheff ^[3] and Kutzbach ^[4]. After Dobrovolski ^[5] proposed the constraint parameter as the mechanism family parameters and introduced the mobility formula, Rössner ^[6] and Antonescu ^[7] simplified the formula, reducing the computational cost. Contributions have continued to emerge in recent decades, and emphasis is placed on common overconstrained linkages, including Bennett linkages and Bricard linkages. Various approaches to Bennett linkage DOF calculation were presented in literature by Chen Yan and Zhong You ^[8] and Eddie Baker ^[9]. Additionally, the calculation of the Bricard linkage's degree of freedom has been addressed through progressively refined theoretical lenses, from computational algebra methods of Arponen, Piipponen and Tuomela ^[10] and the resultant approach of Lewis and Coutias ^[11], to the recent half Bricard linkage concept by Yao et al. ^[12].

Different from the progress of DOF theory of Bennett and Bricard's linkage mechanism, the idea of screw theory as a supplementary constraint mechanism has appeared in recent years. Rico and Ravani ^[13] used group theory to explain the paradox of motion with "displacement conjugation". And recently, the research on using screw theory is divided into two directions. Su et al. ^[14] attached importance to the "duality of constraint and freedom" of rigid bodies and soft mechanisms, Shvalb and Medina ^[15] used the geometry of ruled-surface, the structure of stationary hypo-paradoxical and the dynamic Bennett mechanism, which were not overlapped, but gathered together, revealing the richness of geometry and algebra behind the over-constrained spatial linkage action. By summing up a lot of theoretical processes, this review found a common theme. That is, from the "phase-based constraint number method" to the "framework considering geometry", we finally understand that "easy movement is not a scalar invariant, but a structural property produced by the interaction between constraint space and free space".

2. EVOLUTION OF MOBILITY THEORY FROM 19TH CENTURY TO RECENT

2.1 Evolution and Inherent Limitations of Classical Mobility

In the historical story just now, we traced the establishment of the DOF theory. In this part, we will take a good look at their ideas and their original limitations. In 1884, Grübler published the basic formula ^[1] for a planar mechanism with one degree of freedom:

$$3m - 2p - 4 = 0 \quad (1)$$

Where m is the number of all parts of the mechanism, including fixed element and key elements. This formula accords with most of the facts of ordinary planar mechanical motion. This paper laid the basic framework of the theoretical system and laid the foundation for the future theoretical development. However, this formula only supports one planar mechanical motion of degree of freedom of constraint counting and several planar mechanical motion of degree of freedom, lacking sufficient generality to encompass multiple degrees of freedom planar mechanical motion and spatial mechanical motion.

Based on the pioneering work of Mr. Grübler ^[1], Mr. Koeings ^[2] introduced a new concept. This is a parameter used to define the number of independent constraint equations given by joints and the relative positions of mechanism elements. The formula is as follows:

$$M = 6(m - 1) - C \quad (2)$$

Where c is the number of independent constraint equations given by joints.

Malytshe ^[3] and Kutzbach ^[4] extended this theory to spatial mechanisms and made great progress. Malytsheff published the somov-malytshe formula.

$$M = 6n - \sum_{i=1}^5 iC_i \quad (3)$$

Furthermore, Kutzbach proposed the mobility formula.

$$M = (6 - d)(m - 1) - \sum_{i=1}^p (6 - d - f_i) \quad (4)$$

Generalizing the Somov-Malytsheff formula to all the mechanisms with same order for each independent loop.

Dobrovolski ^[5] defined the constraint parameter as the mechanism family and introduced it to mobility formula:

$$M = (6 - d)(m - 1) - \sum_{i=d+1}^5 (i - d)C_i \quad (5)$$

Based on the comprehensive DOF calculation theory, Rössner ^[6] defined the kinematic chain containing at least one link which have more than three joints as “half open” kinematic chain and verified “half open” kinematic chains had fractioned freedom.

In the last decade of the 20th century, Antonescu ^[7, 16] proposed the concept of average constraint parameters d_a and introduced it to the mobility formula

$$d_a = \frac{1}{q} \sum_{i=1}^q d_i \quad (6)$$

$$M_{da} = (6 - d_a)(m - 1) - \sum_{i=1}^5 (i - d_a)C_i \quad (7)$$

Antonescu ^[7, 16] reduced the computational burden of Dobrovolski 's formula ^[5] by expressing it in a more compact form that required fewer parameter evaluations.

Despite decades of continuous efforts to establish universal mobility theory, the DOF calculation and applicability of these formulas remain limited. The persistent challenge posed by overconstrained mechanisms has exposed the lack of further mathematical theories being incorporated into DOF theory. The Bennett linkage and the Bricard linkage serve as the two iconic and normal examples of overconstrained mechanisms. The following section surveys the most recent advances in the DOF analysis of these two linkages.

2.2 Recent Studies in Two Iconic Linkages: Bennett and Bricard Linkages

To illustrate the diverse methodological responses to limit identified above, two mechanisms of broad applicability are presented below: Bennett and Bricard linkages.

2.2.1 Bennett Linkages

Bennett linkage is the single-loop spatial mechanism composed of four links and four revolute joints, whose DOF equals to -2, based on the traditional Kutzbach-Grübler formula. However, in view of its special geometric conditions, the Bennett linkage has one DOF. Shortly after the beginning of the 21st century, Chen and You [8] verified that a large network consisting of multiple Bennett-linkage loops, connected in specific geometric conditions, still has one DOF.

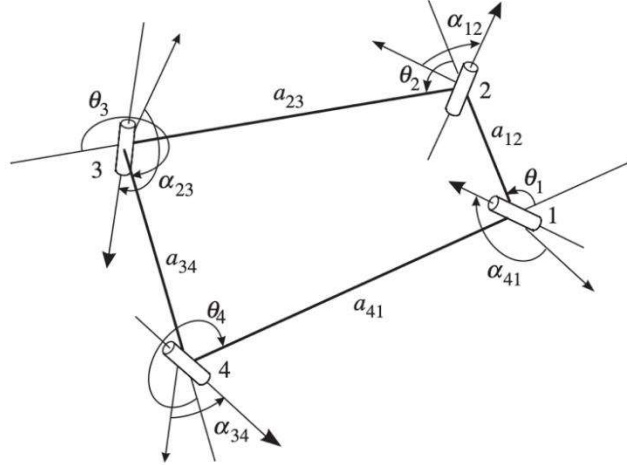


Figure 1. A Bennett linkage [8].

Figure 1 shows a Bennett linkage where the links of specific length are connected in specific twist.

$$a_{12} = a_{34} = a \quad a_{23} = a_{41} = b, \quad (8)$$

$$\alpha_{12} = \alpha_{34} = \alpha \quad \alpha_{23} = \alpha_{41} = \beta, \quad (9)$$

The lengths and twists satisfy.

$$\frac{\sin \alpha}{\sin \beta} = \frac{a}{b} \quad (10)$$

$$\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} = \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\beta - \alpha}{2}} \quad (11)$$

These limitations ensure that only one independent revolute variable exists in the mechanism, accordingly, the linkage has a single degree of mobility.

As shown in Figure 2, the Bennett linkages can be repeated to form a large deployable structure, which only has a DOF, resulting in a high degree of deployability and great stability in other dimensions. This research has been extensively adopted in manufacturing, establishing a connection between DOF theory and structural engineering and demonstrating practical value of DOF theory. Following Chen and You's initial contribution in 2005 [8], they introduced the concept of negative length and proposed three new structures of Bennett linkages in 2008 [17]. This paper pioneered introducing mathematic concept and approach to DOF theory. Based on the concept of Bennett networks, Baker [9] adopted the methodology of repeating linkages to form a large structure, but applied it to a different research subject, doubly collapsible octahedron. The skew network of octahedron is showed in Figure 3. In this paper, the doubly collapsible octahedron was confirmed as identical with members of an established trio of hybrid six-bars. The similarity led to one DOF of the skew network in deploying process.

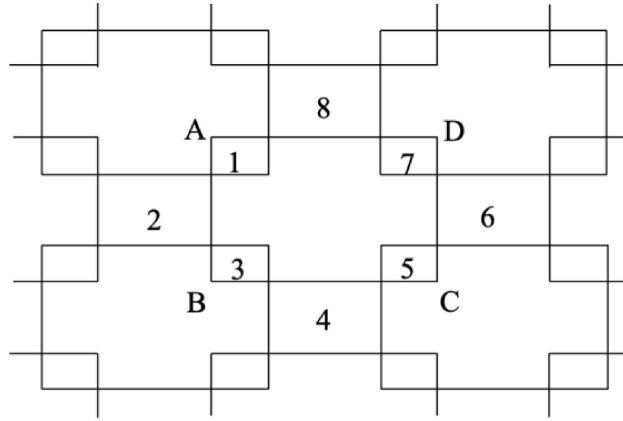


Figure 2. Single-layer network of Bennett linkage [8].

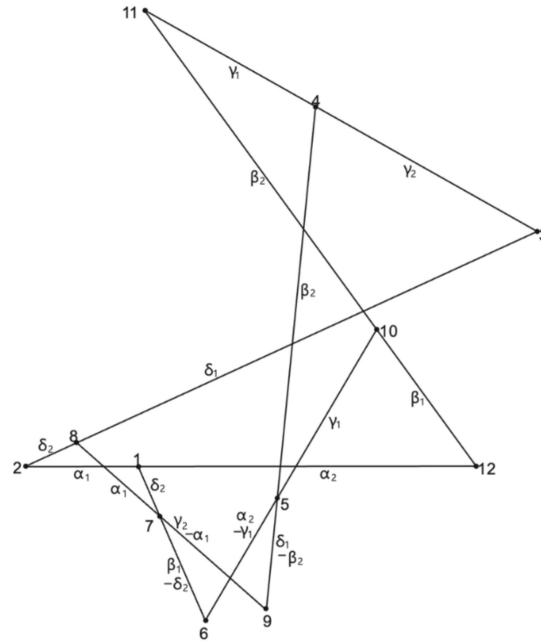


Figure 3. The skew network corresponding to the octahedron [9].

In the paper published in 2024 [18], a kinematic model of the 8R mechanism was established using screw theory. By equivalently disconnecting the mechanism into two open chains, the Jacobian matrix was derived, and the singularities and degrees of freedom were analyzed. It also discussed a series of derivative mechanisms that preserve the same kinematic characteristics as the original 8R linkage. This research revealed the new insights for systematic design for Bennett derivative mechanisms.

Although the 21st century witnessed the introduction of mathematical concepts and tools into the research of Bennett linkages' DOF theory, which significantly expanded its range of applications, the theoretical framework had largely stayed at the level of investigation in specific cases. These mathematical methods provided a new insight for DOF theory. Nevertheless, the development of a systematic and axiomatic theory with a universally applicable framework remains an ongoing challenge, requiring sustained future efforts.

2.2.2 Bricard Linkages

A Bricard linkage is a classical overconstrained single-loop kinematic chain consisting of six links connected by six revolute joints, which, despite violating the Kutzbach-Grübler mobility formula, possesses one degree of freedom due to special geometric conditions. It is a classical overconstrained mechanism. The classical formula gives zero mobility for

this mechanism, whereas the actual behavior shows one DOF. In contradistinction to Bennett linkages, there exist a wide variety of types in Bricard linkages, including line-symmetric [19], plane-symmetric [19] and trihedral [20].

In the early 21st century, tools of computational and mathematical analysis were used in DOF calculation theory. In 2008, Arponen, Piipponen and Tuomela [10] proposed an approach to calculating the mobility of a mechanism by determining the dimension of its configuration space through Gröbner basis and prime ideal decomposition of polynomial constraint equations. This paper created a new DOF calculation of Bricard linkages with polynomial equations, which departed from the conventional mobility formulas and provided new perspectives for future development of mobility theory.

By employing resultant-based elimination and computer algebra, Lewis and Coutsias [11] directly characterized the global solution space of the closure equations, thereby providing a rigorous algebraic criterion for the existence of mobility in overconstrained linkages. Lopez-Custodio [21] focused on line-symmetric Bricard linkages. This paper revealed the kinematic branch of Bricard linkages and their possible kinematic trace, through analysis of two concentric toroid's crossing line in geometry. The approach to analyzing Bricard linkages in this paper differs markedly from Lewis and Coutsias's paper in 2016 [11]. In Lewis and Coutsias's paper [11], they solved systems of highly nonlinear multivariable polynomial equations through matrix and resultant and obtained a table of relations satisfying specific constraints on the parameters. On the other hand, the method proposed by Lopez-Custodio [21] utilized differential geometry to analyze known geometric curves and obtained geometrically visualized kinematic branches and evolution pathways. This paper provides a pioneering insight for research in Bricard linkages, not limited by calculating DOF through algebra, and study further in the geometric structure of kinematic branches. Additionally, it serves as a typical example of applying differential geometry to the reconfiguration analysis of overconstrained mechanisms. Although the methods in Lewis and Coutsias's paper [11] needed to solved high degree polynomial equations with high computational complexity, they were validated on all kinds of Bricard linkages, which had broad applicability. In contrast, the method of Lopez-Custodio [21] avoided complex algebraic calculation. Nevertheless, it just was validated on line-symmetric Bricard linkages. These two approaches are therefore complementary rather than competitive: the algebraic method ensures generality, whereas the geometric method provides efficiency and intuitive insight. Future research will extend the geometric framework to other Bricard types, so that the advantages of two paradigms can be together.

Lu [22] published a new progress in the geometric method of analyzing Bricard linkages, which clarified the conditions of bifurcations of Bricard linkages. This can be interpreted as the continuous intersection of ellipses (including the case of degeneration). Moreover, in this paper, the symmetrical Bricard linkage is bifurcation Point, and the research by Myard 5R linkages, Bennett 4R linkages, spherical 5R linkages, planar 5R linkages and single revolute joint provides a new perspective for the evolution and degradation of institutions. Mathematically unified the essential relationship between Bricard linkages and other overconstrained mechanisms, and promoted the evolution of geometric information framework in DOF theory.

Generally speaking, these results show that algebraic method provides a strict foundation for global mobility analysis, while the ability of geometry-informed method in reducing the calculation cost and revealing the fundamental mechanical structure more clearly is constantly improving. We hope that through the continuous integration of these complementary methods, we will have a more unified understanding of overconstrained mechanisms.

2.3 The Screw-Theoretic Paradigm: From Constraint Counting to Ruled-Surface Geometry

Rico and Ravani [13] put forward new ideas. They reconsidered the problem of mobility under the framework of Lie group theory. In this paper, the concept of group theory conjugation, especially the conjugate transformation of rotations and screws, is used to assemble symmetric linkage mechanisms. After making full use of mathematics, they found many incredible links such as Bennett linkages, partners of R-C-R-C mechanisms, H-C-H-C mechanisms, etc. This study shows the mobility of these linkages. In addition, the geometric parameter conditions of each link motion and the relationship between motions are clarified, which opens up a new way to systematically design and classify overconstrained linkages.

With the systematic development of screw theory, its application to the mobility analysis of overconstrained mechanisms has become increasingly widespread. An instantaneous screw motion is represented by a twist $\hat{T}$. A wrench $\hat{W}$ represents a force-moment system associated with a constraint and is reciprocal to an admissible twist [14].

$$\hat{T} = \begin{Bmatrix} \Omega \\ V \end{Bmatrix} = \begin{Bmatrix} \Omega \\ c \times \Omega + p\Omega \end{Bmatrix} \quad (12)$$

$$\widehat{W} = \begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{Bmatrix} F \\ c \times F + qF \end{Bmatrix} \quad (13)$$

Where p and q are called pitches of twist and wrenches. $\widehat{T}$ and $\widehat{W}$ meet the reciprocal requirement:

$$\widehat{T} \circ \widehat{W} = \Omega \cdot M + V \cdot F = 0 \quad (14)$$

As shown in Figure , this paper described the geometric relationship among all joints with Denavit-Hartenberg parameter and utilized matrix to obtain the precise screw system rank of Waldron six-bar linkage, which is a one DOF overconstrained mechanism composed of two Bennett linkages. The incorporation of screw theory provided a rigorous mathematic framework for DOF analysis of overconstrained mechanisms. This approach went beyond DOF calculation, and it identified the directions of motion and constraint.

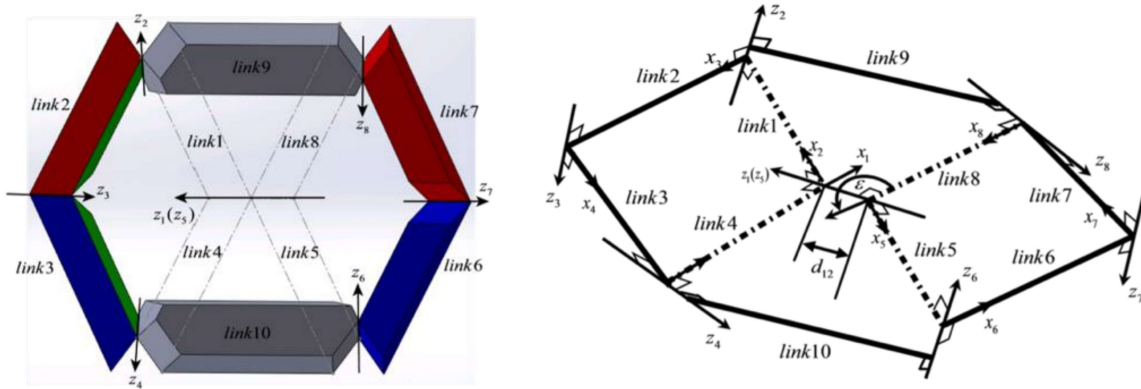


Figure 4. The Waldron six-bar linkage is formed by two Bennett four-bar linkages ^[14].

Recently, in the development mobility theory, the intersection between screw theory and other mathematical methods has become increasingly strong. A recent study (2026) published in Mechanism and Machine Theory has applied screw system rank to determining the linear dependence of axes. According to the linear dependence, the real flexible mechanism and rigid mechanism are separated. In this paper, screw theory serves as a “geometry-algebra bridge”, which transferred the position and direction of joints into unified mathematical objects.

3. DISCUSSION AND CONCLUSION

This review reviews the changes of DOF theory from the past to the recent algebra, geometric and screw-theoretic frameworks. These new frameworks are formulated to solve the problem of overconstrained mechanisms that has been difficult to solve. Throughout the historical process, it is not difficult to find an important contradiction: it is easy to calculate the number of constraints in the previous formula, but the constraints of joints are all different. This decision doesn't work in the most interesting mechanisms. Bennett and Bricard's linkages are examples of this limit. Both are moving, but the previous formula is not moving, or the predicted DOF is negative. So, I began to use difficult math tools. Chen, You and Baker said that Bennett linkages can build a large deployment network working in DOF, and link the DOF theory with structural engineering. The same is true for various types of Bricard linkages, which need various methods, such as Gröbner bases, resultant elimination, Differential-Geometric Branch Analysis, etc.

A great turning point in this development is the systematic use of screw theory. Rico, Ravani and Su gave up the idea that mobility is “a number derived only from shape” and reconsidered “the nature from constraint and free space shape”. The duality of twist-wrench gives us a method to seriously investigate the movable direction and the restricted direction. Recently, the research of using screw theory with ruled-surface geometry and group theory has begun even under the very difficult circumstances of hypo-paradoxical linkages and bifurcations. However, there are still several important topics. There is no one method that can be used for all kinds of institutions, each of which has the trouble of assumption and calculation. Especially in the mechanism with bifurcations and kinematotropic behavior, the difference between limited mobility and micro mobility needs further theory. It is more difficult to make mobility use these methods instead of compliant mechanisms of ideal joints, and screw theory has just begun to explore. In the future, the future of DOF theory will depend more and more on putting multiple mathematical frameworks together: algebraic geometry for global solution space, differential geometry for branch analysis of motion, screw theory for instantaneous constraint and free duality. The

recent research on the combination of screw system rank and ruled-surface geometry is an example of this synergistic effect. This is related to the understanding of more unified mobility beyond the limits of any method. Through such interdisciplinary synthesis, we may eventually approach the standard of general mobility that we have been looking for for a long time. It is not so much a single closed formula as a unified theoretical framework, which can systematically deal with the overall complexity of mechanical constraints and motion.

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